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FINITE ELEMENTS FOR EXTERIOR PROBLEMS USING INTEGRAL EQUATIONS

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SUMMARY

We present some integral methods for exterior problems for the Laplace equation. Then we give finite element approximations for these equations and some errors estimates. Finally, we indicate how these integral equations can be coupled with a usual finite element method on a bounded domain to solve an exterior non-linear problem which is linear far away.

KEY WORDS Finite element method Integral equations Coupling

INTRODUCTION

Let us consider the simplest problem in aeronautics, which is the flow outside an aeroplane in the linear case.

We search for the speed of the air, $\mathbf{u}$ which satisfies

$$\left.\begin{aligned} \operatorname{div}\mathbf{u} &= 0, \quad \text{in} \quad \Omega, \\ \operatorname{curl}\mathbf{u} &= \mathbf{0}, \quad \text{in} \quad \Omega, \\ \mathbf{u}\cdot\mathbf{n}/\Gamma &= 0, \\ \mathbf{u}-\mathbf{u}_\infty &\to \quad \text{at} \quad \infty. \end{aligned}\right\} \tag{1}$$

It is usual in this case to look for the new unknown φ such that

$$\mathbf{u}-\mathbf{u}_\infty = \operatorname{grad}\varphi. \tag{2}$$

This is valid at least when $\mathbf{u}_\infty$ is constant. We obtain

$$\left.\begin{aligned} \Delta\varphi &= 0, \quad \text{in} \quad \Omega, \\ \frac{\partial\varphi}{\partial n} &= \mathbf{u}_\infty\cdot\mathbf{n} \quad \text{on} \quad \Gamma, \\ \operatorname{grad}\varphi &\to 0, \quad \text{at infinity.} \end{aligned}\right\} \tag{3}$$

Problem (3) has a unique solution which tends to zero at infinity. Moreover as we have

$$\int_\Gamma \mathbf{u}_\infty\cdot\mathbf{n}\,\mathrm{d}\gamma = 0, \tag{4}$$

Based on an invited lecture

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the solution φ is such that

$$\left.\begin{aligned} |\varphi| &\leqslant \frac{c}{r^2}, \\ |\nabla\varphi| &\leqslant \frac{c}{r^3}, \end{aligned}\right\} \text{ when } r \to \infty. \tag{5}$$

We can now represent this potential φ using either a *simple layer potential or a double layer potential.*

THE SIMPLE LAYER REPRESENTATION

The auxiliary unknown is in this case the jump of $\partial\varphi/\partial n$ across Γ where the interior problem has been chosen as the solution of the Dirichlet problem. Let us call it q. We have

$$\varphi(\mathbf{y}) = \frac{1}{4\pi} \int_\Gamma \frac{q(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\gamma(\mathbf{x}). \tag{6}$$

The integral equation coming from (6) and the boundary condition is then

$$\mathbf{u}_\infty \cdot \mathbf{n}(\mathbf{y}) = -\frac{q(\mathbf{y})}{2} + \frac{1}{4\pi} \int_\Gamma q(\mathbf{x}) \frac{\partial}{\partial n_y} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}\gamma(\mathbf{x}). \tag{7}$$

This is an equation of Fredholm type with a weak singularity. It has a unique solution.

Using this representation can lead to an ill-conditioned problem; for instance suppose that the surface Γ is quite flat, as shown in Figure 1.

In the limit case the *simple layer potential remains continuous, but the solution φ is not continuous.*

THE DOUBLE LAYER REPRESENTATION

This representation will be valid when the surface Γ is flat. The auxiliary unknown is the jump α of φ across the surface Γ. The associated interior problem is

$$\left.\begin{aligned} &\Delta\varphi = 0, \\ &\frac{\partial\varphi}{\partial n} = \mathbf{u}_\infty \cdot \mathbf{n}, \quad \text{on} \quad \Gamma, \end{aligned}\right\} \tag{8}$$

a solution of which is known in this case because $\mathbf{u}_\infty$ is constant:

$$\varphi|_{\text{int}} = (\mathbf{u}_\infty \cdot \mathbf{x}). \tag{9}$$

The function φ has the expression

$$\varphi(\mathbf{y}) = -\frac{1}{4\pi} \int_\Gamma \alpha(\mathbf{x}) \frac{\partial}{\partial n_x} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}\gamma(\mathbf{x}). \tag{10}$$

We have

Figure 1.

$$\alpha = \varphi|_{\text{ext}} - (\mathbf{u}_\infty \cdot \mathbf{x}). \tag{11}$$

It is possible to write *two integral equations* satisfies by α. The usual one is obtained by the boundary condition (9) and is

$$(\mathbf{u}_\infty \cdot \mathbf{y}) = \frac{\alpha(\mathbf{y})}{2} - \frac{1}{4\pi}\int_\Gamma \alpha(\mathbf{x}) \frac{\partial}{\partial n_x}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \mathrm{d}\gamma(\mathbf{x}). \tag{12}$$

This equation is of Fredholm type with a weak singularity. It has a unique solution and is the adjoint of equation (7) above.

The second equation is obtained by using the boundary condition

$$\frac{\partial \varphi}{\partial n} = \mathbf{u}_\infty \cdot \mathbf{n}, \quad \text{on} \quad \Gamma. \tag{13}$$

The trouble comes when computing the corresponding kernel

$$\frac{\partial^2}{\partial n_x \, \partial n_y}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right),$$

which appears to be hypersingular. We introduce in that case a variational formulation.[1]

To α defined on Γ, we associate

$$\mathrm{curl}_\Gamma \alpha = \mathbf{n} \wedge \mathrm{grad}\, \alpha. \tag{14}$$

The variational formulation is then

$$\frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{(\mathrm{curl}_\Gamma \alpha(\mathbf{x}) \cdot \mathrm{curl}_\Gamma \beta(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\gamma(\mathbf{x})\, \mathrm{d}\gamma(\mathbf{y}) = \int_\Gamma \mathbf{u}_\infty \cdot \mathbf{n} \beta \, \mathrm{d}\gamma, \quad \forall \beta \in H^{1/2}(\Gamma)/R. \tag{15}$$

The above bilinear form is symmetric and coercive, i.e. there exists positive c such that

$$c\,\|\alpha\|^2_{H^{1/2}(\Gamma)/R} \leqslant \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{(\mathrm{curl}_\Gamma \alpha \cdot \mathrm{curl}_\Gamma \alpha)}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\gamma(\mathbf{x})\, \mathrm{d}\gamma(\mathbf{y}). \tag{16}$$

Now we have two different equations for the same unknown and the same problem.

APPROXIMATION

The usual approximation for the equation (12) is the collocation method.

The surface Γ is approximated by a union of triangles (or rectangles). The function α is then approximated by a constant on each triangle and equation (12) is collocated at one point of each triangle.

This approximation is quite poor. It is not clear that it is *stable*. Moreover we need to compute $\nabla_\Gamma \varphi$, i.e. $\nabla_\Gamma \alpha$. This is an unstable operation starting from α constant on each element.

We can give a variational formulation for equation (12) which takes the form

$$\int_\Gamma \alpha\beta \, \mathrm{d}\gamma - \frac{1}{4\pi}\int_\Gamma\int_\Gamma [\alpha(\mathbf{x}) - \alpha(\mathbf{y})]\beta(\mathbf{y}) \frac{\partial}{\partial n_x}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \mathrm{d}\gamma(\mathbf{x})\, \mathrm{d}\gamma(\mathbf{y})$$
$$= \int_\Gamma (\mathbf{u}_\infty \cdot \mathbf{y})\beta \, \mathrm{d}\gamma, \quad \forall \beta \in L^2(\Gamma). \tag{17}$$

To such a formulation, we can easily associate an approximated problem by introducing a finite element approximation of the space $L^2(\Gamma)$. For instance when using the classical P_1 element

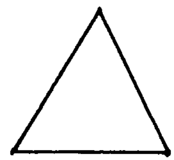

Figure 2.

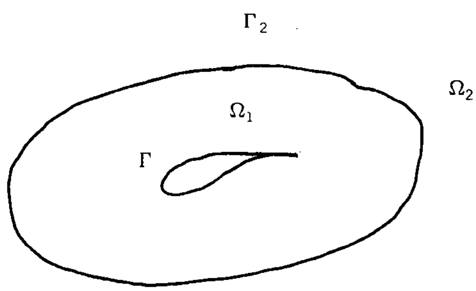

Figure 3.

(Figure 2) for both the surface Γ_h and the space V_h the approximate problem becomes[2]

$$\int_{\Gamma_h} \alpha_h \beta_h \,\mathrm{d}\gamma - \frac{1}{4\pi}\int_{\Gamma_h}\int_{\Gamma_h} (\alpha_h(\mathbf{x}) - \alpha_h(\mathbf{y}))\beta_h(\mathbf{y})\frac{\partial}{\partial n_x}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\gamma(\mathbf{x})\,\mathrm{d}\gamma(\mathbf{y})$$
$$= \int_{\Gamma_h} (\mathbf{u}_\infty \cdot \mathbf{y})\beta_h \,\mathrm{d}\gamma, \quad \forall \beta_h \in V_h. \tag{18}$$

In this case, we can prove error estimates of the form

$$|\alpha - \alpha_h \psi|_{L^2(\Gamma)} \leqslant ch^2$$
$$|\varphi(\mathbf{y}) - \varphi_h(y)| \leqslant \underbrace{ch^2}_{\text{geometry}} + \underbrace{ch^4}_{\text{space } V_h} \quad \text{for } \mathbf{y} \notin \Gamma \tag{19}$$

We can also associate with equation (15) an approximated problem. With the same finite element as above, we obtain

$$\frac{1}{4\pi}\int_{\Gamma_h}\int_{\Gamma_h} \frac{(\mathrm{curl}_{\Gamma_h}\alpha_h(\mathbf{x}_h)\cdot \mathrm{curl}_{\Gamma_h}\beta_h(\mathbf{y}_h))}{|\mathbf{x}_h - \mathbf{y}_h|}\mathrm{d}\gamma_h\,\mathrm{d}\gamma_h$$
$$= \int_{\Gamma_h} \mathbf{u}_\infty \cdot \mathbf{n}\beta_h \,\mathrm{d}\gamma, \quad \forall \beta_h \in V_h/R. \tag{20}$$

We can also prove some error estimates in this case:[2]

$$|\alpha - \alpha_h \psi|_{L^2(\Gamma)} \leqslant ch^2 + ch \quad \text{(normal approximation)},$$
$$|\varphi(\mathbf{y}) - \varphi_h(\mathbf{y})| \leqslant \underbrace{ch^2 + ch}_{\text{geometry}} + \underbrace{ch^4}_{\text{space}}, \quad \mathbf{y} \notin \Gamma.$$

COUPLING VIA INTEGRAL EQUATIONS

Suppose now that we have a problem in an exterior domain which is non-linear, but linear (or almost) far enough away from the body.

For instance, take a transonic flow problem, far enough away the density is almost constant. In that case, the equation becomes linear and of the type considered above.

We introduce then an artificial boundary Γ_2 (see Figure 3).

In terms of the potential φ, the equation is now replaced by a system:

$$\left.\begin{aligned} &A(\varphi_1)=f, \quad \text{in} \quad \Omega_1,\\ &\left.\frac{\partial\varphi_1}{\partial n}\right|_{\Gamma}=\mathbf{u}_\infty\cdot\mathbf{n},\\ &\Delta\varphi_2=0, \quad \text{in} \quad \Omega_2,\\ &[\varphi]=0,\\ &\left[\frac{\partial\varphi}{\partial n}\right]=0, \end{aligned}\right\} \text{ across } \Gamma_2. \tag{21}$$

We then choose some auxiliary unknowns:

$$\left.\begin{aligned} \omega&=\varphi_1|_{\Gamma_2}=\varphi_2|_{\Gamma_2},\\ \lambda&=\left.\frac{\partial\varphi_1}{\partial n}\right|_{\Gamma_2}=\left.\frac{\partial\varphi_2}{\partial n}\right|_{\Gamma_2}. \end{aligned}\right\} \tag{22}$$

The problem in Ω_1 has a variational formulation of the form

$$\int_{\Omega_1} B(\varphi_1)\psi\,\mathrm{d}x-\int_{\Gamma_2}\lambda\psi\,\mathrm{d}\gamma=\int_{\Omega_1} f\psi\,\mathrm{d}x, \quad \forall\psi. \tag{23}$$

If we decide to represent the solution of the problem in Ω_2 using a double layer potential, the equation linking the unknowns α and λ is given by (15).

We also have

$$\omega(\mathbf{y})=-\frac{\alpha(\mathbf{y})}{2}-\frac{1}{4\pi}\int_\Gamma \alpha(\mathbf{x})\frac{\partial}{\partial n_x}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\gamma(\mathbf{x}). \tag{24}$$

We thus arrive at the system

$$\begin{aligned} &\int_{\Omega_1} B(\varphi_1)\psi\,\mathrm{d}x-\frac{1}{4\pi}\int_{\Gamma_2}\int_{\Gamma_2}\frac{(\mathrm{curl}_\Gamma\alpha\cdot\mathrm{curl}_\Gamma\psi)}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\gamma\,\mathrm{d}\gamma=\int_{\Omega_1} f\psi\,\mathrm{d}x,\\ &\int_{\Gamma_2}\varphi_1\beta\,\mathrm{d}\gamma+\frac{1}{4\pi}\int_{\Gamma_2}\int_{\Gamma_2}[\alpha(\mathbf{x})-\alpha(\mathbf{y})]\beta(\mathbf{y})\frac{\partial}{\partial n_x}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}\gamma\,\mathrm{d}\gamma=0,\\ &\forall\psi\in H^1(\Omega_2) \quad \text{and} \quad \beta\in H^{1/2}(\Gamma_2)/R. \end{aligned} \tag{25}$$

We can associate with this system an approximated system using finite elements.

We have to be careful with the compatibility between the choice of finite elements for ψ and β. The choice of P_1 for both is O.K.

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